The development of arithmetical abilities

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Background: Arithmetical skills are essential to the effective exercise of citizenship in a numerate society. How these skills are acquired, or fail to be acquired, is of great importance not only to individual children but to the organisation of formal education and its role in society. **Method:** The evidence on the normal and abnormal developmental progression of arithmetical abilities is reviewed; in particular, evidence for arithmetical ability arising from innate specific cognitive skills (innate numerosity) vs. general cognitive abilities (the Piagetian view) is compared. **Results:** These include evidence from infancy research, neuropsychological studies of developmental dyscalculia, neuroimaging and genetics. The development of arithmetical abilities can be described in terms of the idea of numerosity – the number of objects in a set. Early arithmetic is usually thought of as the effects on numerosity of operations on sets such as set union. The child's concept of numerosity appears to be innate, as infants, even in the first week of life, seem to discriminate visual arrays on the basis of numerosity. Development can be seen in terms of an increasingly sophisticated understanding of numerosity and its implications, and in increasing skill in manipulating numerosities. The impairment in the capacity to learn arithmetic – dyscalculia – can be interpreted in many cases as a deficit in the concept in the child's concept of numerosity. The neuroanatomical bases of arithmetical development and other outstanding issues are discussed. **Conclusions:** The evidence broadly supports the idea of an innate specific capacity for acquiring arithmetical skills, but the effects of the content of learning, and the timing of learning in the course of development, requires further investigation. **Keywords:** Arithmetic, cognitive development, dyscalculia, numerosity, number, infants, child.

'A child, at birth, is a candidate for humanity; it cannot become human in isolation.' (Pieron, 1959)

Numerosity as the basis of arithmetic

The child acquiring arithmetical skills in our kind of numerate society encounters a variety of number-specific cultural tools. The most obvious are the numerical expressions: number words (*one, two, twenty-two, million...*), numerals (1, 2, 22, 1000000...), roman numerals, patterns on dice, cards and dominoes. Others will be relatively abstract: arithmetical facts ($5 \times 3 = 15$), arithmetical procedures (borrowing, long multiplication), arithmetical laws ($a + b = b + a$; if $a-b = c$ then $a = b + c$; and so on). The skills that need to be acquired include reading and writing numbers, counting objects in a set, calculating in the four basic arithmetical operations, reading numerals aloud, writing numerals, applying these skills in money tasks, telling time and dates, finding a page in a book, selecting a TV channel, and so on. All of these skills are much more complex and subtle than they may at first appear to competent adults. The question addressed in this review is this: is the process of acquiring these tools of arithmetic supported only by general-purpose cognitive capacities – such as reasoning, memories (short- and long-term), and a sense of space – or are we born with number-specific capacities?

The child acquiring number skills is probably not helped by the fact that a numerical expression does not have a single meaning. 'Four' (or '4') can, for example, be the name of a TV channel – a quite arbitrary signifier, rather like a proper name or label. It can also be a page number, and be part of a familiar fixed sequence, coming immediately after page 3 and before page 5. Numbers are also used to refer to continuous analogue quantities, such as '4.6 grams'. These uses are not unique to numerical expressions: TV channels can be named with words or acronyms (Fox, ABC); letters of the alphabet form a familiar fixed sequence; and other quantity expressions (yard, ton) have been used for millennia. (For a more detailed account of the 'situations' in which numerical expressions are used, see Fuson, 1988.)

The distinctive meaning of numerical expressions is to denote the number of things in a set – the *numerosity* of a set. (The term 'numerosity' is used here as the cognitive counterpart to the term 'cardinality' used by mathematicians and logicians). This is what is special about numbers. It entails that numerosity is abstract: it is not a physical object or the property of an object (such as a colour or shape). Rather, it is a property of a set that can have any type of member: physical objects, sounds, or other abstract objects (as in three wishes). This does not make numerosities mysterious or a convenient fiction (Giaquinto, 2001), but it does mean that our grasp of particular numerosities may depend on the nature of the sets. For example, the numerosity of a familiar patterns of dots, such as are seen on dice, is easier to apprehend than the same number of dots randomly arranged, and it gets more difficult the more dots there are (Mandler & Shebo,1982).

The idea of numerosity entails or embodies familiar consequences such as two sets have the same numerosity if and only if members of each can be put in one-to-one correspondence with none left over. Broadly, a child will understand the concept of numerosity if she or he:

- understands the one-to-one correspondence principle;
- understands that sets of things have numerosity and that some manipulations of these sets affect the numerosity – combining sets, taking subsets away, and so on – and that one set has the same numerosity as another, or a greater numerosity, or a smaller numerosity;
- understands that sets need not be of visible things; they can equally be audible things, tactile things, abstract things (like wishes);
- can recognise small numerosities – sets of up to about four objects – without verbal counting.

(following Butterworth, 1999)

Now, the child has to sort out when a numerical expression is being used as a label, to locate an object in a sequence, to refer to an amount of stuff or as a definite numerosity.

The usual arithmetical operations of addition, subtraction, multiplication and division can be defined in terms of operations on sets and their numerosities, and this is how we normally think about them. The sum of an addition, for example, can be thought of as the numerosity of the union of two, or more, disjoint sets; similarly, subtraction, multiplication and division can be thought of in terms of the results of operations on sets (Giaquinto, 1995). In formal curricula, multiplication, division and fractions typically follow addition and subtraction, and are explained in terms of them. For example, the mathematics curriculum in the UK begins in Reception (4–5 yrs) and Year 1 (5–6 yrs) with counting, adding and subtracting, and then in Year 2 introduces 'the operation of multiplication as repeated addition or as describing [e.g., counting] an array' (DfEE, 1999, Key Objectives, p. 3), and table facts are taught up to Year 5. Fractions are introduced in Year 3, and division, as the complement of multiplication, in Year 4.

This review will focus on counting, addition and subtraction, since these are the topics which are the most researched, and which are the pedagogical, and to a large extent, the conceptual basis of other aspects of arithmetic. However, multiplication, division and fractions involve concepts that are not readily derivable from the concept of numerosity. This is discussed further below.

One of the key debates is whether the child is helped to understand the special numerosity meaning by possessing a specific innate capacity for numerosities, rather than, say, a capacity for dealing with, or being sensitive to, quantities more generally. Crucial evidence comes from the people who appear to have a selective deficit in this capacity which profoundly affects their ability to learn arithmetic. This condition is known as 'dyscalculia', and is discussed in detail below.

Although it is widely agreed that possession of something like the concept of numerosity is necessary for normal arithmetical competence, it is by no means agreed how individuals arrive at this concept. According to Piaget (1952), necessary preconditions were a grasp of certain logical principles, since arithmetic is really a part of logic:

> Our hypothesis is that the construction of number goes hand-in-hand with the development of logic, and that a pre-numerical period corresponds to a pre-logical level. Our results do, in fact, show that number is organised, stage after stage, in close connection with gradual elaboration of systems of inclusion (hierarchy of logical classes) and systems of asymmetrical relations (qualitative seriations), the sequence of numbers thus resulting from an operational synthesis of classification and seriation. In our view, logical and arithmetical operations therefore constitute a single system that is psychologically natural, the second resulting from a generalisation and fusion of the first, under two complementary headings of inclusion of classes and seriation of relations, quality being disregarded. (Piaget, 1952, p. viii)

Thus, for Piaget, our idea of numerosity was built on more basic capacities. These included the capacity to reason transitively; that is, the child should be able to reason from the facts that if A is bigger than B, and B is bigger than C, then A is bigger than C. Without this capacity, the child could not put the numbers in order of size, which is clearly fundamental. A second capacity that the child must develop is the idea that the number of things in a set is 'conserved', to use his technical term, unless a new object is added to the set, or an object subtracted from it. Merely moving the objects around – should not affect number: for example, spreading them out so they take up more room. Even more basic than either of these two capacities, as Piaget pointed out, is the ability to abstract away from the perceptual properties of the things in the set. To grasp the numerosity of a set, one needs to ignore all the particular features of the objects in it: their colour, their shape, their size, and even what they are: a set of three cats has the same numerosity as a set of three chairs, or indeed of three wishes. The idea of number is abstract. And the ideas of the 'same number' or 'different numbers' are abstractions from abstractions. The emergence of the capacity for numerosity will depend on the development of the necessary prior capacities, what Piagetians call 'prerequisites'. It will also depend, as do many conceptual and logical abilities, on interacting with the world. The concept of numerosity could emerge as a result of manipulating objects, for example, lining up sets to establish one-to-one correspondence between the members of the two sets, for sharing out sweets or toys.

Some authors have proposed that cognitive capacities that are not specific to number are necessary, or at least, very important, in acquiring arithmetical skills. These include working memory (e.g., Ashcraft, Donley, Halas, & Vakali, 1992; Hulme & Mackenzie, 1992), spatial cognition (e.g., Rourke, 1993), and linguistic abilities (e.g., Bloom, 1994; Carey & Spelke, in press). Correlations between these cognitive abilities and standardised tests of arithmetic are well established. For example, Bull and Johnston (1997) found a correlation of −.54 between one measure of language ability (naming latencies) and maths achievement. However, it is by no means clear how the causal relationships work: does good arithmetic help on working memory, spatial or language tasks? Is there a common factor that underpins all of the tasks? It is possible to explore the causal relationships systematically in children with specific deficits in arithmetic learning and with impairments to the putative supporting cognitive capacities.

Infant capacities

Contrary to what Piaget and others have proposed, infants seem to respond to the numerical properties of their visual world, without benefit of language, abstract reasoning, or much opportunity to manipulate their world.

Numerosity detection/recognition and manipulation

In a pioneering experiment, Starkey and Cooper (1980) showed that 4–6-month-olds were sensitive to the numerosity of an array of black dots using a 'habituation–dishabituation' paradigm. Infants like novelty and will look longer at new things. The same thing repeatedly causes them to habituate, lose interest, while a new thing causes them to regain interest – to dishabituate. In this study, infants would dishabituate to a new number of dots up to four. Of course, with each change of numerosity, there will be changes of other stimulus dimensions, including total amount of blackness, total length of edge, and perhaps spatial frequency. Starkey and Cooper tried to control for this by changing the arrangement of the habituation dots in each trial, and by ensuring that the dishabituation stimulus covered the same extent.

In a study of children of 6–8 months, Starkey, Spelke, and Gelman (1990) used pictures of objects, such as an orange, a keychain, sunglasses, a glove and so on. Instead of cards with two dots close together alternating with two dots far apart, each card had two objects, but different objects each time. Therefore, each new card with the same number of objects had new pictures on it. The dishabituating card also had new pictures on it, but there were three pictures this time. Since each card was new for the

baby, would the baby have mentally categorised the habituating cards as showing two things, so that when a card with three things was presented, they would regain interest and look longer? If they do look longer, it cannot be because of mere novelty, since each card was new. It turned out that babies did look significantly longer at the card with three pictures on it. Again, the infants seemed to be sensitive to the number of pictures on the card. This means they categorised what they saw in a way that is quite abstract: the particular features of each picture – the colour, the objects depicted, their size, their brightness – which change with every card – have to be disregarded.

Similar findings have been reported for babies in the first week of life (Antell & Keating,1983). Van Loosbroek and Smitsman (1990) showed babies of 5 and 13 months 2, 3 or 4 rectangles in shades of grey that moved in random trajectories on a computer monitor. From time to time one rectangle would appear to pass in front of another, occluding part of it. As in the previous studies, after a while the babies looked at the screen less, but when the number of rectangles changed, either by adding one more rectangle, or taking one away, they started to look significantly more. They cannot have been responding to a change in the pattern, since each of the rectangles was in constant motion, so they must have extracted the numerosity from the moving displays.

Infants also seem not only to recognise small numerosities up to about 4, they also seem to have some sense of their relative size. Brannon (2002) showed 11-month-old infants a sequence of displays with an increasing number of dots, and then a test sequence. If this sequence had a *decreasing* number of dots, the infant would look about twice as long as if it also had an increasing number. However, infants two months younger did not show this effect.

Despite these demonstrations of infants' sensitivity to numerosities, several studies have sought to show that infants are responding to continuous quantity rather than numerosity. Certainly, when these are put into conflict rather than controlled or randomised, continuous quantity seems a more powerful cue (Feigenson, Spelke, & Carey, 2002; Mix, Huttenlocher, & Levine, 2002). However, a recent study using groups of moving dots, where continuous quantity of figural area and contour are strictly controlled, showed that infants do respond to numerosity (Wynn, Bloom, & Chiang, 2002).

There is, of course, no reason why infants should not have brain systems for processing both types of stimulus. Normally, the environment correlates the two types – more objects will normally have greater spatial extent – so that the outputs of two systems will be consistent, so relying on continuous quantity as a guide to numerosity could be adaptive. More experience of the world, and stress on the importance of number, would change the way these conflicts are resolved.

Is there an upper limit to the infant's concept of numerosity? Can she enumerate 4, 10, or 100? Three seems to be the maximum, though infants in the Starkey and Cooper (1980) study distinguished 4 from 3, but 4 for them may have represented just 'more than three'. However, we cannot be sure that this limitation lies in the baby's idea of numerosity rather than in her ability to perceive and to remember what has been perceived. Our understanding that numerosities have no limit seems to depend on our sense that it is always possible to keep adding one. Thus, any limitation on the infant's part could have more to do with her ability to carry out successive additions, and the chain of reasoning needed to get from that to the idea numbers have no upper limit.

The most likely limitation is the ability to take in the numerosity of visual array of objects at a glance, and without counting. Even in adults, the limit is about four. This seems to be a specialised process in visual perception, which is usually given the name 'subitising' (Mandler & Shebo, 1982). Dehaene and Changeux (1993) have created a computer model of this process, which very simply and effectively extracts the number of objects from a visual display, disregarding their size, shape or location. The representation that is extracted can then be trained to make comparisons. It is tempting to think that something like this has been built into the visual processing system of the infant's brain. For numerosities beyond four, infants dishabituate when there is a 2:1 ratio (e.g., 8:16) but not when there is a 2:3 ratio (8:12) (Xu & Spelke, 2000).

Possessing a concept of numerosity implies more than just being able to decide whether two sets do or do not have the same numerosity. It implies an ability to detect a change in numerosity when new members are added to the set, or old members are taken away – in other words, to be able to compute the arithmetical consequences on adding and subtracting. Wynn (1992) showed that infants are able to do this, making use of the fact that babies look longer at events that violate their expectations. Infants of 4 to 5 months were shown a doll being placed on a stage, then covered by a screen, and then a second doll placed behind the screen. The infant could now see no dolls at all and had to imagine the situation behind the screen. If the infant had computed that one doll plus one doll makes two dolls, then her *arithmetical expectation* would be that there would be two dolls behind the screen. Wynn found that when the screen was removed, infants looked longer when there was one doll or three dolls than when there were two dolls. Similarly, when two dolls were placed on the stage, covered, and one doll shown to be removed, infants expected that there would be one doll left, and looked longer at other numbers.

This experiment has now been frequently replicated, and Simon, Hespos, and Rochat (1995) have shown that 3–5-month-old infants look longer when the number of dolls is unexpected than when their identity is unexpected (i.e., one doll is surreptitiously changed behind the screen). There is also evidence that infants are responding to numerosity rather than location (Koechlin, Naccache, Block, & Dehaene, 1999).

These studies have not been without their critics, and there have been failures to replicate (Wakeley, Rivera, & Langer, 2000), and alternative explanations in terms of familiarity of the objects displayed (Cohen & Marks, 2002). Wynn has replied to both critiques, noting differences in experimental procedure that could have led to different outcomes (Wynn, 2000, 2002).

More radically, Carey, Spelke, and their colleagues have suggested that the mental operations that seem to involve numerosities can really be explained in terms of two essentially non-numerical processes. First, there is an object-tracking system that is needed in any case to maintain attention to up to four objects in the environment. Experiments which demonstrate that infants respond to changes in number, or indeed, to changes from the expected number, are explicated in terms of changes in the state of the object-tracking system. Second, there is a system for representing and comparing continuous quantity (Carey & Spelke, in press). In the experiment by Xu and Spelke (2000) described above, the discontinuity between minimum ratio for small number discrimination (2:3) and large number discrimination (1:2) is explained in terms of a shift from the object-tracking system to the continuous quantity system. (Apprehension of exact numerosities greater than 4 depend, according to this view, on acquiring number words.)

However, the current balance of evidence favours the idea that infants are able to represent the numerosity of sets of objects and carry out mental manipulations over these representations.

Development of counting

One of the earliest and perhaps the most important contact between the child's sense of number and the conceptual tools provided by the culture is counting. Many nursery rhymes involve counting or counting words (One, two buckle my shoe, On the first day of Christmas), and even the titles of stories for children contain number words (Snow White and the Seven Dwarves, The Famous Five). Counting is complex skill which involves learning the counting words in the correct order, coordinating the production of counting words with the identification of objects in the set to be counted, and that each object in the set is counted once and only once. Moreover, the child has to understand that the process of counting can yield the number of objects in the set.

The philosopher John Locke (Locke, 1690/1961) recognised that counting words are helpful in keeping in mind distinct large numerosities. 'Some

Americans I have spoken with (who were otherwise of quick and rational parts enough) could not, as we do, by any means count to 1,000; nor had any distinct *idea* of that number, though they could reckon very well to 20.' These Americans, the Tououpinambos from the Brazilian jungle, 'lacked names for numbers above 5'. Locke believed that we construct the idea of each number from the idea of 'one' ('the most universal idea we have'). By repeating 'this idea in our minds and adding the repetitions together … thus by adding [the idea of] of one to the [the idea of] one, we have the complex idea of a couple.' He thought that number names were essential for acquiring distinct ideas of largish numbers, and that a *system* of number names 'conduce[s] to well reckoning' (Locke, 1690/1961). Thus for Locke, the basic ideas of numerosity are available to us without the help of culture, but that culture can be helpful in some circumstances.

Of course, Locke depended on for this conclusion on casual observation rather than systematic investigation and it would certainly be extremely interesting to use modern methods to explore this hypothesis in children raised in cultures that lacked names for numbers above 5. A few such cultures still exist, in Amazonia, New Guinea and notably in Australia where few of the Aboriginal language have number words above three, and those come through borrowing (Dixon, 1980).

Learning the counting words

Counting makes the first bridge from the child's innate capacity for numerosity to the more advanced mathematical achievements of the culture into which she was born. The least mathematical of cultures enable their members to do much more than the infant. They can keep track of quite large numerosities counting with special number words or body-part names; they can do arithmetic beyond adding or subtracting one from small numerosities which they will need for trading or for ritual exchanges.

Though it seems very to easy to us adults, learning to count takes about four years from two to six. Children start around two years old, progress in stages until about 6 years old when they understand how to count and how to use counting in a near-adult manner.

Gelman and Gallistel (1978) have identified the skills, what they call 'principles', that are required to be able to count. Consider the example of a child counting five dinosaurs:

- The number words from 'one' to 'five', or, more properly, we need to know five counting words that we always keep in the same order. (The 'stable order principle'.)
- Each of these words must be linked with one and only one object: no word must be used more than once and all objects must be counted. That is, we must put each object in one-to-one correspondence with the counting words. (the 'one-to-one principle'.)
- The child must be in a position to announce the number of toy dinosaurs by using the last counting word used: 'One, two, three, four, five. Five toy dinosaurs.' (The 'cardinal principle'.)

Gelman and Gallistel (1978) proposed two further principles, 'abstractness', which means that anything can be counted, and 'order-irrelevance', which means that you can start counting with any object in the set. It is clear that a grasp of the principles follows from understanding the concept of numerosity. Sets are not intrinsically ordered. Understanding this means that you understand the order-irrelevancy principle. There is also no constraint on the kinds of things that can be members of a set, provided they can be individuated. Understanding this implies holding the principle of abstractness. Of course, children, and adults, may possess the concept of numerosity without fully understanding and without having derived all the principles that validly follow from it.

Learning the sequence of counting words is the first of these skills mastered. Children seem to know at about two and half what a number word is, and rarely intrude non-number words into the sequence, even when the order is incorrect (Fuson, 1988, Chapter 10).

Even learning the sequence of number words is not that straightforward. Children of two or three years often think of the first few number words as just one big word 'onetwothreefourfive' and it takes them some time to learn that this big word is really five small words (Fuson, 1992). Gelman and Gallistel's (1978) observation of a $3\frac{1}{2}$-year-old child trying to count eight objects show that getting the sequence right is a difficult stage: 'One, two, three, four, eight, ten, eleben. No, try dat again. One, two, three, four, five, ten, eleben. No, try dat again. One! two! three-ee-four, five, ten, eleben. No. … [finally] … One, two, three, four, five, six, seven, eleven! Whew!'

One-to-one correspondence appears at about two years of age quite independently of learning the sequence of counting words. At 2, children are able to give one sweet to each person, put one cup with each saucer and can name each person in a room or a picture, or point to them, once and only once (Potter & Levy, 1968). If you show a 'puppet who is not very good at counting' counting the same object twice or missing an object altogether, children of $3\frac{1}{2}$ are very good at spotting these violations of one-to-one correspondence (Gelman & Meck, 1983). And almost all children point to each object when they count, even when they can use the number words correctly, so there is one-to-one correspondence between objects, points and words (Gelman & Gallistel, 1978).

Children of three years or so may count in some but not all appropriate circumstances. When asked

to give three toy dinosaurs, they may just grab a handful and give them to you without counting. Wynn (1990) calls them 'Grabbers'. Grabbers clearly know that number words represent a set of more than one, even if they have not yet grasped the role of number words in counting, and do not use the last word of a count to say how many.

At perhaps an earlier stage they think that the number word is just a *label* that attaches to an object. Here is what Adam, a Grabber, did in one of Wynn's tasks.

> Experimenter (E): So how many are there? Adam (A): [Counting three objects ...] One, two, five! E: [Pointing towards the three items] So there's five here? A: No, that's *five* [pointing to the item he'd tagged 'five'] ... E: What if you counted this way, one, two, five? [Experimenter counts the objects in a different order than Adam has been doing] A: No, *this* is five [pointing to the one he has consistently tagged five]

In a give-a-number task, 'Counters', usually a few months older, will count, either aloud or silently, passing you the toys one by one. They also reliably give you the last word of the count in answer to 'How many?', satisfying the cardinal principle. These children are initially able to count only small numerosities, and probably build up their competence systematically from 1 to 2, from 2 to 3, from 3 to 4 and so on up. In a give-a-number task, they will start by being able reliably to give 1, then to give 2, but perhaps not 3, then 3 but perhaps not 4. So by $3\frac{1}{2}$ most children have grasp of small numerosities, and know that counting is a way to find the numerosity of a set.

According to Gelman and her colleagues, children learning to count know the principles before their skills are fully developed (e.g., Gelman & Gallistel, 1978). Certainly, children's performance is affected by number size, with larger numbers being harder (e.g., Fuson, 1988), and mastery of the three principles is not completely synchronised, with stable-order being reliably earliest, one–one correspondence between counting words and objects following later, and the cardinal principle the last of the three (Fuson, 1988, Chapter 10).

The cardinal word principle – the last number named in a count is the numerosity of the set counted – also follows from the concept of numerosity, since you are establishing a correlation between members of a set whose numerosity you do know, the number words up to five, say, and members of the set of things to be counted, whose numerosity you do not know. It may follow in a practical way as well. Recall that infants can recognise the numerosities of objects up to about 3.

Fuson (1988) suggests that children may notice that when they count a set 'one two three', they get the same number as when they subitise the set. This helps them realise that counting up to *N* is a way of establishing that a set has *N* objects in it. Repeating the count, and getting the same number obtained from subitising, will reinforce the idea that every number name represents a unique numerosity. Again, this is something obvious to us, but may not be obvious to the child, especially as in practice the child will sometimes count the same set and get different results. He will count (or miscount) 'one two three dinosaurs', and may count again, 'one two four dinosaurs', and then again, 'one two three four dinosaurs'. He may wonder whether different number words can name the same numerosity, the numerosity of the set of dinosaurs.

Piaget (1952) was among the first to see that full grasp of the concept of numerosity meant being able to abstract away from – ignore – irrelevant perceptual features of the set to be enumerated, so that you do not think, for example, there are more things just because they are more spread out (or more closely packed together). He saw the development of the child's thinking in general as a move away from the particular to the general and abstract.

The conflict between the different sources of evidence about numerosity can be seen very clearly in the way children between 4 and 6 try to establish whether two sets have the same number. What seems to happen is that during this period, they come to relegate perceptual cues such as the spacing of objects, and to depend exclusively on genuine numerosity information, such as correspondence and counting. They cease to be fooled by changing the spacing of objects. In Piagetian terms, number is 'conserved' under perceptual transformations. The child progresses to conservation, the sign that the number concept is grasped, in stages. First, the child relies solely on perceptual cues; then the child will be able to use one-to-one correspondence but may still rely more on perceptual cues, and finally, the child will rely entirely on correspondence, and will not be fooled by perceptual cues.

Piaget believed that counting, and learning number words to do it, was not necessary to construct the concept of numerosity, which he thought was built up from logical concepts and reasoning until possession of the concept was evidenced by conservation of number under transformations at about 6 years.

Development of arithmetic

Counting is the basis of arithmetic for most children. Since the result of adding two numerosities is equivalent to counting the union of two disjoint sets with those numerosities, children can learn about adding by putting two sets together and counting the members of their union.

From counting all to counting on

Children make use of their counting skills in the early stages of learning arithmetic. The number words, as

was noted in the Introduction, have both a sequence and a numerosity (or cardinal) meaning. As Fuson and Kwon (1992) point out, 'In order for number words to be used for addition and subtraction, they must take on cardinal meanings' (p. 291). Children often represent the numerosity of the addend by using countable objects, especially fingers, to help them think about and solve arithmetical problems.

There appear to be three main stages in the development of counting as an addition strategy:

1. Counting all. For 3 + 5, children will count 'one, two, three' and then 'one, two, three, four, five' countables to establish the numerosity of the sets to be added, so that two sets will be made visible – for example, three fingers on one hand and five fingers on the other. The child will then count all the objects.
2. Counting on from first. Some children come to realise that it is not necessary to count the first addend. They can start with three, and then count on another five to get the solution. Using finger counting, the child will no longer count out the first set, but start with the word 'Three', and then use a hand to count on the second addend: 'Four, five, six, seven, eight'.
3. Counting on from larger. It is more efficient, and less prone to error, when the smaller of the two addends is counted. The child now selects the larger number to start with: 'Five', and then carries on 'Six, seven, eight'.

(Butterworth, 1999; Carpenter & Moser, 1982)

The stages are not strictly separate, in that children may shift strategies from one problem to the next. There is a marked shift to Stage 3 in the first six months of school (around 5–6 years in the US, where this study was conducted (Carpenter & Moser, 1982). Stage 3 shows a grasp of the fact that taking the addends in either order will give the same result. This may follow from an understanding of the effects of joining two sets, that is, taking the union of two disjoint sets.

Even in the earliest phases of the development of addition abilities, children do not need to count the union of the sets. In one set of experiments, Starkey and Gelman (1982) showed the children two sets one at a time so that there was no opportunity to count all the elements. In these circumstances, most three-year-olds could solve 2 + 1, and a few could solve 4 + 2. By 5 years, all could solve the first and 81% the second. Interestingly, only 56% solved 2 + 4, suggesting that some of the children were not counting on from larger, but were still counting on from first.

From counting on to arithmetical facts

The skilled adult typically will not need to calculate or count single digit problems such as 3 + 5, 3 × 5, 5 − 3, or 6 ÷ 3 and will simply retrieve the solution from memory.

A variety of models of the mental organisation of arithmetical facts has been proposed. One influential view has been that children learn to associate 3 + 5 with several answers, but the association with 8 will end up as the strongest (Siegler & Shrager, 1984). Another view is that facts are typically stored as specifically verbal associations, though subtraction and division require further processes of 'semantic elaboration' involving manipulation of an analogue magnitude representation (Dehaene & Cohen, 1995). In both models, retrieval will depend on the learning history of the individual. Thus, facts that are learned earlier or practised more will show greater accessibility.

The single strongest argument against these views is that retrieval times show a very strong *problem-size effect* for single-digit problems: the larger the sum or product the longer the problem takes to solve (Ashcraft et al., 1992). This factor is much more potent than frequency of occurrence (see Butterworth, Girelli, Zorzi, & Jonckheere, 2001).

Note also that children who are using a counting strategy to solve arithmetic problems are not using memory retrieval. It is likely that memories are laid down during Stage 3 of counting on from larger. This would mean that the child would work out the result of Larger Addend + Smaller Addend (rather than First Addend + Second Addend) and store it in that form. Some evidence for this comes from Butterworth et al. (2001), who showed that adults, who presumably retrieve answers, are quicker to solve Larger Addend + Smaller Addend problems than Smaller Addend + Larger Addend problems. The frequency of the problems in textbooks was not a good predictor of solution times. Both this and the problem-size effect suggest that addition facts are organised in terms of number size rather than as orthogonal verbal vectors or a network of associations modulated by practice effects.

Similar results were obtained for children, 6–10 years old, doing multiplication. Larger × Smaller was faster than Smaller × Larger, even though the (Italian) education system taught Smaller × Larger earlier. For example, 2 × 6 is in the (Italian) 2× table which is taught before 6 × 2, which is in the 6× table (Butterworth, Marchesini, & Girelli, 2003). In fact, this study showed that children start by privileging the form in which the problem is taught, and later reorganise their memory store to privilege the Larger × Smaller format. Again, this suggests a specifically numerical organisation to arithmetical facts. They are not just rote associations.

Multiplication, division and fractions

Curricula typically introduce multiplication and division later than addition and subtraction, and explain them in terms of repeated addition and repeated subtraction and partitions of sets, thus building on concepts of sets and numerosities.

Indeed, Piaget's (1952) treatment of multiplication is in terms of one-to-many correspondence some years later than addition and subtraction. However, ideas of division as sharing are actually very early in development, in some respects earlier even than counting (see Nunes & Bryant, 1996), and the idea of a half as the partition of a set is introduced in Year 1 of school in the UK (DfEE, 1999). Some multiplication problems can certainly be solved by addition or by double counting the multiplier and the multiplicand.

However, thinking about multiplication or division of two numbers just in terms of sets with one–many correspondences fails to do justice the kinds of situation the child encounters in everyday life, as well as in the classroom. Prices such as '50p each' is neither the set of objects to be bought nor the monetary set of the cost. Rather it is a relationship between the two sets, and remains the same (is conserved, if you will) whether six objects are bought or sixty. So multiplication by the price does not increase the price, but only the cost. The price is a ratio, or a kind of division, which is conserved under some types of multiplication. For example, 2/4 is the same ratio as 4/8 or 100/200. Understanding this is fundamental to understanding a whole range of primary school mathematics, including multiplication, division, and fractions.

These kinds of numbers are often referred to as 'intensive quantities', to distinguish them from numbers whose meanings are 'extensive', that is, sets (Schwartz, 1988). Interpreting numbers as intensive quantities is needed for everyday problems involving temperature and concentration. Children of 6 to 8 believe that if you add two cups of water, each at 40°C, the resulting mixture will be warmer than the originals, because you are adding temperatures (Stavy & Tirosh, 2000); and children of 10 to 11 find it hard to work out which of two mixtures of orange juice concentrate and water will taste more orangey: 3 cups of concentrate to 2 cups of water, or 4 cups of concentrate to 3 cups of water (Noelting, 1980a, 1980b). In neither type of case does a grasp of numerosity fully prepare the child to reason in the appropriate way. Piaget (1952) noted that problems involving proportions would be difficult.

Division also introduces a new type of number in terms of fractions and decimals, namely, rational numbers. These will only have been encountered previously in the concept of a half, but they are important in the everyday context of measures. Again, concepts entailed by numerosities (such as each number has a unique successor) will not work in these contexts.

Nunes and Bryant (1996), in a very useful review, begin their discussion of multiplicative reasoning with the caution: this 'is a very complicated topic because it takes different forms and it deals with many situations, and that means that the empirical research on this topic is complicated too' (p. 143). It seems clear that where the child can think about multiplication and division as manipulations on sets, then it is relatively easy to acquire, but when the task demands grasp of numbers as intensive quantities then it is difficult.

Understanding arithmetical concepts

Children enter school with informal concepts of number and arithmetic based on their experiences of counting and calculation; however, much educational practice was, and still is, focused on drilling basic arithmetical facts such as number bonds and tables. The theoretical justification came from the work of Thorndike (1922), formulator of the 'law of effect' – or what we would now call reinforcement – which stated that associations that lead to 'satisfying states of affairs' are reinforced, those that lead to unsatisfying states weakened. The idea then was to build networks of reinforced associations between number combinations such as 5 + 3 and their arithmetical result. As the network, carefully constructed by the teacher, is built in the mind of the child, so the generalisations (concepts and laws) would be grasped. Of course, Thorndike insisted that drilling the facts had to be fun, which meant, among other things, being able to see their practical applications. More recently, the 'distribution of associations' model (Siegler, 1988; Siegler & Shrager, 1984) has been influential. Here it is assumed that the child may associate a number combination with both the wrong and the correct answer. The key to arithmetical success is to strengthen the association with the right answer. The model predicts that the performance on single digit arithmetical fact tasks will be the relative frequency of the association between the problem (e.g., 6 + 3, 6 × 3, 6 − 3, 6 ÷ 3) and the correct solution (9, 18, 3, 2) as compared with the frequency of association between the problem and incorrect solutions.

Even as the Thorndike approach was being taken up by educators, an alternative was being pursued by Brownell, who advocated 'meaningful learning' rather than drill (Brownell, 1935). Although research showed that drill can make retrieval of facts faster, transfer of learning to new problems was much better with meaningful learning. (See Resnick & Ford, 1981, Chapter 1, for a discussion.) The time course of developing an understanding of arithmetical concepts and principles, and applying them in a meaningful way, is thus likely to be heavily influenced by the educational practices the child undergoes (Canobi, Reeve, & Pattison, 1998).

Commutativity, associativity

The role of understanding has been tested on commuted pairs of addition facts (6 + 3, 3 + 6) and multiplication facts (6 × 3, 3 × 6).

If commutativity is understood, then is it necessary or even desirable to store, in long-term memory, both forms of the commute? There is evidence that the form with M + n is accessed more readily than n + M (Butterworth et al., 2001). This does not, however, entail understanding. It may just mean that the child has learned that it does not matter which order the addends appear.

As mentioned above, Butterworth et al. (2003) found that children of 6 to 10 years of age learning multiplication tables reorganise their memories to privilege M × n, over n × M, even when n × M was learned earlier and presumably practised more. Again, this suggests, though does not prove, that these children understand the commutativity of multiplication.

Interestingly, some cultures do not teach the whole set of multiplication facts from 1 × 1 to 9 × 9 in tabular form. In China, they only teach one half of the set, beginning with 2 × 2 (the 1 × table being considered trivial) to 2 × 9; but since 2 × 3 has already been learned, the 3 × table begins with 3 × 3, and so on. In this way, only 36 facts have to be acquired, and the equivalence of the commuted pairs has to be learned (Yin Wengang, personal communication). Research shows that Chinese adults are more accurate and quicker at solving multiplication problems than their Western peers (Campbell & Xue, 2001; LeFevre & Liu, 1997). Although this has been attributed to more drill (Campbell & Xue, 2001; Penner-Wilger, Leth-Steensen, & LeFevre, 2002), it may reflect exactly the opposite – fewer facts to memorise and better understanding.

Complementarity

Piaget (1952) has argued, quite reasonably, that a child does not really understand addition or subtraction without understanding the relationship between them. That is, if 5 + 3 equals 8, then 8 – 5 must equal 3, and 8 – 3 must equal 5. This is the Principle of Complementarity. All this should follow from an understanding of sets and numerosities: if set B is added to set A, and then removed, the resulting numerosity will still be A.

Do children understand the Principle of Complementarity, and if so at what age or stage does this understanding begin? Now, of course, it is perfectly possible to arrive at the correct answer without understanding the Principle of Complementarity. Conversely, it is possible to understand the principle, yet sometimes get the answer wrong. This means that the ability or inability to solve these problems is not a sure guide to understanding. Rather, investigators have asked whether 'inversion' problems that can be solved by the principle are solved better than control problems that cannot. Starkey and Gelman (1982) found no convincing evidence for understanding in children of 3 to 5 years of age, while other researchers have found evidence of understanding in older children (Stern, 1992).

A systematic study of this issue was recently reported by Bryant, Christie, and Rendu (1999). They looked at 5–7-year-olds, and carefully controlled for types of solution strategies that might be used. For example, in a task using a set of objects, if three new objects are added, and then exactly the same are taken away, then the correct answer may be achieved on the basis of a general 'undoing' procedure that could apply to non-numerical situations such as splashing paint on a wall and washing it off. Bryant et al. controlled for this by comparing adding and removing the same objects with adding and removing the same number of different objects. They also looked at equivalent problems with numerals. Children were much more successful with inversion problems, such as 12 + 9 – 9, than control problems matched for sum, such as 10 + 10 – 8. What is more, they could use the Principle in more complex problems that required decomposition of the subtrahend. Thus, they appeared to make use of the Principle in problems such as 7 + 4 – 5 by decomposing 5 into 4 + 1. Indeed, many of the children revealed by analysis of performance to be using the principle were able to state it in words, but by no means all.

Although the children who used the principle to solve inversion problems did better overall than those who calculated the solutions, by no means all the children who did well used the principle. Factor analyses and correlations revealed two separate factors: a calculating factor and an understanding factor. Similar issues arise in connection with complementarity of multiplication and division. If 9 × 3 = 27 is known, then 27 ÷ 9 = 3 and 27 ÷ 3 = 9 should both follow without the need for calculation.

Table 1 summarises the principal milestones in the normal development of arithmetic.

Sex differences in arithmetic?

In academic achievement, boys have in the past outperformed girls by the age of 18. This has been an official worry since the Cockcroft *Committee of Inquiry into the Teaching of Mathematics* produced its report for the British government. But that was in 1982. Before then, few seemed to care, and many thought it almost improper for girls to be good at maths. In a relatively enlightened *Handbook for Teachers,* issued in 1937, the British government advised:

> In mental capacity and intellectual interests [boys and girls] have much in common, the range of difference in either sex being greater than the difference between the sexes. But in early adolescence the thoughts of boys and girls are turning so strongly towards their future roles as men and women that it would be entirely inappropriate to base their education solely on their intellectual similarity. (See Cockcroft (1982), Appendix B)

Table 1 Milestones in the early development of arithmetic

Age	Milestones (Typical study)
0;0	Can discriminate on the basis of small numerosities (Antell & Keating, 1983)
0;4	Can add and subtract one (Wynn, 1992)
0;11	Discriminates increasing from decreasing sequences of numerosities (Brannon, 2002)
2;0	Begins to learn sequence of counting words (Fuson, 1992); can do one-to-one correspondence in a sharing task (Potter & Levy, 1968)
2;6	Recognises that number words mean more than one ('grabber') (Wynn, 1990)
3;0	Counts out small numbers of objects (Wynn, 1990)
3;6	Can add and subtract one with objects and number words (Starkey & Gelman, 1982); Can use cardinal principle to establish numerosity of set (Gelman & Gallistel, 1978)
4;0	Can use fingers to aid adding (Fuson & Kwon, 1992)
5;0	Can add small numbers without being able to count out sum (Starkey & Gelman, 1982)
5;6	Understands commutativity of addition and counts on from larger (Carpenter & Moser, 1982); can count correctly to 40 (Fuson, 1988)
6;0	'Conserves' number (Piaget, 1952)
6;6	Understands complementarity of addition and subtraction (Bryant et al, 1999); can count correctly to 80 (Fuson, 1988)
7;0	Retrieves some arithmetical facts from memory

However, at the time of writing (2004) girls in England easily outperform boys in all subjects at all ages. There is one exception to this general rule: mathematics. Girls are only just outperforming boys. (DfES, 2004)

Geary (1996) reviewed a wide range of industrialised countries to show that boys, on average, still outperform girls in mathematical problem solving. Among US teenagers, there are more boys than girls in the upper reaches of the SAT-M (Scholastic Aptitude Test – Mathematics), a requirement for university admission. The difference between boys and girls gets larger the higher up the range.

However, even in the USA at 17 years the *average* difference between boys and girls is still only 1%. The most recent cross-national comparisons using the same tests in all countries, the Third International Maths and Science Survey (TIMSS, Keys, Harris, & Fernandes, 1996) reinforces the overall picture that in most countries, including the USA, there is no statistical difference between boys and girls (see Table 2).

However, there are still a few countries in which boys reliably outperform girls, most dramatically in England. What is certainly clear from the TIMSS data is that the differences between countries, between educational practices, has a vastly greater effect on performance than the difference between sexes.

Developmental dyscalculia

Disorders of numeracy development, 'developmental dyscalculia' (henceforth DD), can prove useful in understanding the course of normal development and addressing our original question: is dyscalculia the consequence of general-purpose cognitive capacities or is it due to an abnormalities of the in an innate capacity for numerosities?

DD has been defined by the UK Department for Education and Skills as:

A condition that affects the ability to acquire arithmetical skills. Dyscalculic learners may have difficulty understanding simple number concepts, lack an intuitive grasp of numbers, and have problems learning number facts and procedures. Even if they produce a correct answer or use a correct method, they may do so mechanically and without confidence. (DfES, 2001)

This definition draws attention to the 'intuitive grasp of numbers' which is essentially grasping the idea of numerosities. The other problems faced by dyscalculic learners stem from the lack of an intuitive grasp of number.

Prevalence of disorders of learning arithmetic

Specific disorders of numeracy are neither widely recognised nor well understood. Children can be bad at maths in many different ways. Some may have particular difficulty with arithmetical facts, others with procedures and strategies (Temple, 1991), while most seem to have difficulties across the whole spectrum of numerical tasks (Landerl, Bevan, & Butterworth, 2004). Traditional definitions (e.g., DSM-IV) state that the child must substantially underachieve on a standardised test relative to the level expected given age, education and intelligence, and must experience disruption to academic achievement or daily living. Standardised attainment tests, however, generally test a range of skills, which may include spatial and verbal abilities, before collapsing the total into one global score of 'maths attainment'. In addition, standardised tests are diverse, so what is meant by 'maths attainment'

Table 2 International comparisons of sex differences in arithmetical abilities at two ages (from Keys et al., 1996)

Age 9–10			Age 14		
Country	Mean (points)	Difference in favour of boys	Country	Mean (points)	Difference in favour of boys
Singapore	625	−10	Singapore	601	0
Scotland	520	0	Hungary	502	1
USA	544	2	Canada	494	2
Canada	533	3	Germany	485	2
Hungary	549	5	Scotland	464	3
England	513	5	USA	476	5
Norway	502	5	Sweden	478	5
Japan	693	8*	France	493	8
Netherlands	577	15*	Japan	571	11*
			Switzerland	506	14*
			England	476	17*

*Statistically significant difference.

may vary substantially between tests. For this reason it has been hard for researchers to pinpoint the key deficits in dyscalculia, or to be sure how to define dyscalculics for study. A range of terms for referring to developmental maths disability has emerged, including 'developmental dyscalculia' or DD (Shalev & Gross-Tsur, 1993; Temple, 1991); 'mathematical disability' or MD (Geary, 1993); 'arithmetic learning disability': AD, ARITHD, or ALD (Geary & Hoard, 2001; Koontz & Berch, 1996; Siegel & Ryan, 1989); 'number fact disorder' or NF (Temple & Sherwood, 2002); and 'psychological difficulties in mathematics' (Allardice & Ginsburg, 1983). As Geary (1993) and Geary and Hoard (2001) remark, these different classifications seem in most cases to describe the same condition.

The term 'developmental dyscalculia' will be used in this Annotation, but is intended to refer to all these groups. Table 3 shows three population estimates of prevalence.

There are striking differences between these estimates, presumably due to differences in criteria. There is also a striking co-morbidity with deficits in literacy, despite differences among the studies in both criteria and orthography. Nevertheless, more than half of all the dyscalculic children reported in these studies have no literacy deficit. This gives rise to two important issues: (1) why is the incidence of literacy difficulties among dyscalculics and of maths difficulties among dyslexics so high relative to normal children? (2) why are the majority of both groups free of a double deficit?

Characteristics of dyscalculia

It is generally agreed that children with dyscalculia have difficulty in learning and remembering arithmetic facts (Geary, 1993; Geary & Hoard, 2001; Ginsburg, 1997; Jordan & Montani, 1997; Kirby & Becker, 1988; Russell & Ginsburg, 1984; Shalev & Gross-Tsur, 2001), and in executing calculation procedures. Temple (1991) has demonstrated using case studies that these abilities are dissociable in developmental dyscalculia, though this does not seem to be true of the majority of dyscalculic children who have problems with both (Russell & Ginsburg, 1984).

Many researchers suggest that dyscalculia is secondary to more general or more basic cognitive abilities such as semantic memory (Geary et al., 2000, 2001). However, neuropsychological studies of adults with neurological damage strongly indicate that number knowledge is dissociable from semantic memory (Cappelletti, Butterworth, & Kopelman (2001), and that the semantic memory systems for numerical and non-numerical information are localised in different areas of the brain (Thioux, Seron, & Pesenti, 1999).

Working memory difficulties have also been implicated. Geary (1993) suggests that poor working memory resources not only lead to difficulty in executing calculation procedures, but may also affect learning of arithmetic facts. Koontz and Berch (1996) tested children with and without dyscalculia using both digit and letter span (the latter being a measure

Table 3 Prevalence estimates of developmental dyscalculia

STUDY Location	Estimate of learning disability	Criterion	Percentage co-morbid literacy disorder
OSTAD (1998) Norway	10.9% 'Maths disabled'	Registered for special long-term help	51% Spelling disorder
LEWIS et al. (1994) England	3.6% 'Specific arithmetic difficulties'	<85 on arithmetic test, >90 on NVIQ	64% Reading difficulties
GROSS-TSUR et al. (1996) Israel	6.4% 'Dyscalculic'	Two grades below chronological age	17% Reading disorder

of phonological working memory capacity which is not confounded with numerical processing). This study found that dyscalculic children performed below average on both span tasks, though IQ was not controlled. McLean and Hitch (1999) found no difference on a non-numerical task testing phonological working memory (nonword repetition), suggesting that dyscalculic children do not have reduced phonological working memory capacity in general, although they may have a specific difficulty with working memory for numerical information. Temple and Sherwood (2002) found no differences between groups on any of the working memory measures (forward and backward digit span, word span and the Corsi blocks) and no correlation between the working memory measures and measures of arithmetic ability. Thus, although various forms of working memory difficulty may well co-occur with maths difficulties, there is no convincing evidence implicating any form of working memory as a causal feature in dyscalculia.

While there is a high co-morbidity between numeracy and literacy disabilities (see Table 2), it is unclear why this should be. Rourke (1993) has suggested that those suffering a double deficit will have a left hemisphere problem, while the pure dyscalculics will have a right hemisphere abnormality affecting spatial abilities. However, Shalev, Manor, and Gross-Tsur (1997) found no qualitative difference between children with both reading and maths disability and children with maths disability only. No quantitative differences on mathematics tasks were found between dyscalculic children and those with both dyslexia and dyscalculia when the groups were matched for IQ (Landerl et al., 2004).

Other conditions which have been associated with DD are ADHD (Badian, 1983; Rosenberg, 1989; Shalev et al., 2001), poor hand–eye coordination (Siegel & Ryan, 1989); and poor memory for non-verbal material (Fletcher, 1985). Shalev and Gross-Tsur (1993) examined a group of seven children with developmental dyscalculia who were not responding to intervention. All seven were suffering from additional neurological conditions, ranging from petit mal seizures through dyslexia for numbers, attention deficit disorder and developmental Gerstmann's syndrome.

In summary, while it is clearly the case that DD is frequently co-morbid with other disabilities, causal relationships between the disorders have not been proven. In addition, the utility of subtyping dyscalculics according to neuropsychological or cognitive correlates will not be clear until it has been shown that the different subtypes display qualitatively different patterns of numerical deficit.

DD appears to be a specific problem with understanding basic numerical concepts, especially the concept of numerosity. This could affect even very simple tasks such as counting or comparing numerical magnitudes, as was suggested in our

account of normal development. Geary, Hamson, and Hoard (2000) found small but systematic group differences between 1st grade dyscalculic children and controls in magnitude comparison, while Koontz and Berch (1996) found that dyscalculic children appeared to be counting to three rather than subitising in a dot-matching task. Both of these studies suggest that this very fundamental capacity could be tied to the child's understanding of numerosity. Certainly, it has been argued that it underpins the acquisition of counting skills (Fuson, 1988).

One recent study showed reliable reaction time differences between dyscalculic children and maths normal children (including a group with dyslexia) on tests of counting and of number magnitude comparison (Landerl et al., 2004). A specific dyscalculia screener is based on reaction time measures of estimating the number of dots and magnitude comparison (Butterworth, 2003).

Is there a specific neuroanatomical system?

Functional neuroimaging reveals that the parietal lobes, especially the intraparietal sulci, are active in numerical processing and arithmetic (Dehaene, Piazza, Pinel, & Cohen, 2003), and studies of brain-lesioned patients (Cipolotti & van Harskamp, 2001) have identified the left IntraParietal Sulcus (IPS) and the angular gyrus as critical to normal arithmetical performance. Simpler numerical capacities, such as the ability to estimate the numerosity of small sets, appear to be specialised in the right IPS (Piazza, Mechelli, Butterworth, & Price, 2002).

To date, it is not known whether the intraparietal sulci underpin infant capacities, and hence their role in subsequent development is far from clear. However, a recent voxel-based morphometric study of the brains of adolescents with poor arithmetic presents intriguing evidence. Isaacs, Edmonds, Lucas, and Gadian (2001) studied two groups of adolescents with very low birth-weight. One group was cognitively normal, while the second had a deficit just on the numerical operations subtest of the WOND (Wechsler, 1996). When the brains of these two groups were compared, those with arithmetical impairment had less grey matter in the left IPS. Of course, we cannot say whether less grey matter in the left IPS was a cause of poor arithmetic, or its consequence.

Is there a specific genetic basis?

Kosc (1974), in one of the earliest systematic studies of Developmental Dyscalculia (DD), proposed a role for heredity. A recent twin study showed that for DD probands, 58% of monozygotic co-twins and 39% of dizygotic co-twins were also DD and that the concordance rates were .73 and .56, respectively (Alarcon, Defries, Gillis Light, & Pennington, 1997). In a family study, Shalev et al. (2001) found that

approximately half of all siblings of children with DD are also dyscalculic, with a 5–10 times greater risk than for the general population.

Children with Williams Syndrome, who have relatively spared language abilities despite severely impaired cognitive abilities, show abnormalities on simple numerosity tasks such as number comparison, and are also much worse on simple numerical tasks such as seriation, counting, and single digit arithmetic than chronological age- and mental-age matched controls, and children with Down's Syndrome (Paterson, Girelli, Butterworth, & Karmiloff-Smith, submitted).

Some abnormalities of the X chromosome appear to affect numerical capacities more severely than other cognitive abilities. This is particularly clear in Turner's Syndrome where subjects can be at a normal or superior level on tests of IQ, language and reading, but severely disabled in arithmetic (Butterworth et al., 1999; Rovet, Szekely, & Hockenberry, 1994; Temple & Carney, 1993; Temple & Marriott, 1998).

Conclusions

Table 1 summarises the principal milestones in the development of arithmetic by age. There are no age norms for the milestones described here, and the ages are those at which most of the children tested demonstrate these capacities with reasonable reliability. Bear in mind that the studies described are not focused on ages, but on stages; different children can reach the milestones at very different ages.

The milestones are intended to be culture-free, but the data comes from studies of children raised in European and US contexts. There is evidence that the structure of the number word system can speed or slow the acquisition of arithmetical concepts, so children raised in languages with a very regular system, such as Chinese, acquire some arithmetical concepts earlier (Butterworth, 1999; Nunes & Bryant, 1996).

Broadly, then, the development of arithmetic can be seen in terms of an increasingly sophisticated understanding of numerosity and its implications, and in increasing skill in manipulating numerosities. The impairment in the capacity to learn arithmetic – dyscalculia – can be interpreted in many cases as a deficit in the concept in the child's concept of numerosity. It is worth noting, however, that there are several major gaps in our knowledge. The relationship between the earliest capacities shown in the infant and later numerical competencies still needs to be described in detail, especially in regard to the emergence of the specialised left hemisphere brain system. It is also not yet determined whether there is a critical or sensitive period of acquiring arithmetical concepts, and how this might interact with educational input.

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